NOTES ON NORMAL SURFACES IN 4-MANIFOLD TRIANGULATIONS

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In an effort to find knotted surfaces as normal surfaces in the 3-skeleta of triangulated 4-manifolds, we build the machinery to understand normal surfaces in this venue.

1. 3-manifold Motivation

This project has its genesis in the study of normal curves and graphs in 2-complexes and the classification of 2-skeleta in 3-manifold triangulations. In light of the motivation, the 2-complexes considered in this section are exactly those that can be constructed from a finite collection of 2-simplices $\mathbf{\Delta}$, a finite collection of 1-simplices $\mathbf{I}$, and a collection of gluing maps $\mathbf{\Phi} = \{\phi : \sigma_e \to \varepsilon\}$ taking each edge σ_e of a 2-simplex in $\mathbf{\Delta}$ to a 1-simplex $\varepsilon \in \mathbf{I}$ and requiring that each 1-simplex in $\mathbf{I}$ is the image of some $\phi \in \mathbf{\Phi}$. Our 2-complexes are then the identification spaces $S = (\mathbf{\Delta} \sqcup \mathbf{I})/\mathbf{\Phi}$.

A useful restriction we have inadvertently placed on our class of 2-complexes is that no vertex can have a neighborhood homeomorphic to a cone over a collection of disjoint copies of S^1. Said another way, given a regular neighborhood $N(v)$ of a vertex v, $N(v) - v$ is connected. Supposing to the contrary, the completion of each connected component yields a new triangulations of S but with the same edge identifications. Even with this restriction, the 2-skeleta of triangulations of 3- and 4-manifolds remain under consideration.

An arc α in a 2-simplex σ is a *normal arc* if the intersection $\alpha \cap \partial\sigma = \partial\alpha$ and the components of $\partial\alpha$ lie in the interiors of distinct edges of σ. A curve γ embedded in a 2-complex S is a *normal curve* if the lift of $\gamma \cap \sigma$ is a collection of disjoint normal arcs for all σ in $\mathbf{\Delta}$. A graph Γ embedded in S is a *normal graph* if the lift of $\Gamma \cap \sigma$ is a collection of disjoint normal arcs for all σ in $\mathbf{\Delta}$. A normal graph then has all vertices in the interior of the 1-skeleton of S.

Lemma 1.1. *If S is homeomorphic to a surface, then an embedding $\gamma : S^1 \to S$ such that γ does not bound a disk in S is isotopic to a normal curve.*

Venturing away from surfaces, we immediately find problems in the usual normalization procedure. One such example is found in Figure 1. With the intent of studying knotted $(n-2)$-submanifolds, however, this complication is intriguing. One could possibly proceed by loosening the definition of normality by requiring $\gamma \cap \mathrm{Int}(\sigma)$ be a collection of normal arcs. This is contrary to the spirit of the established normal theories and we will instead accept that we will miss some embedded curves that we would otherwise like to capture in this theory.

Consolation for this choice comes quickly.

To Bus

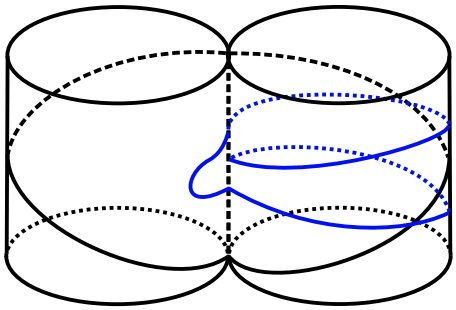


FIGURE 1. Two triangulated annuli joined along an edge. The blue curve is not isotopic to any normal curve despite being essential.

Lemma 1.2. *Let K be a smooth knot in S^3. There exists a Heegard splitting $S^3 = H_1 \cup_\Sigma H_2$ such that K can be moved via ambient isotopy to an essential surface on Σ.*

Proposition 1.3. *For every smooth knot K in S^3 there exists a one-vertex triangulation $\mathscr{T}$ of S^3 such that K is ambient isotopic to a normal curve in the 2-skeleton of $\mathscr{T}$.*

Proof. Let K be a smooth knot in S^3 and let $S^3 = H_1 \cup_\Sigma H_2$ be as in Lemma 1.2. Using layered triangulations [1], Σ can be extended in both directions to one-vertex triangulations of H_1 and H_2 (not necessarily isotopic) so that the attaching map coming from the Heegard splitting $\phi_\Sigma : \partial H_1 \to \partial H_2$ induces an isomorphism of the boundary triangulations (triangulations of Σ). Thus resulting in a one-vertex triangulation $\mathscr{T}$ of S^3 with a triangulation S of Σ found as a subcomplex of $\mathscr{T}^{(2)}$. By Lemma 1.1, K is isotopic to a normal curve in $S \subset \mathscr{T}^{(2)}$. □

2. Oriented Matching Equations

We will begin to consider surfaces embedded in 3-complexes. As in the 2-complex case, let $\mathbf{\Delta}$ be a finite collection of 3-simplices, $\mathbf{F}$ a finite collection of 2-simplices, and a collection of gluing maps $\phi = \{\phi : \Delta_\sigma \to t\}$ taking each face Δ_σ of a 3-simplex in $\mathbf{\Delta}$ to a 2-simplex $t \in \mathbf{F}$ and requiring that each 2-simplex in $\mathbf{F}$ is the image of some $\phi \in \mathbf{\Phi}$. Our 3-complexes are then the identification spaces $M = (\mathbf{\Delta} \sqcup \mathbf{I})/\mathbf{\Phi}$.

To each 3-simplex in $\mathbf{\Delta}$ and each 2-simplex in $\mathbf{F}$ assign an orientation (Figure 2). For each oriented 2-simplex t in M and each $\Delta \in \mathbf{\Delta}$ with face Δ_σ mapping to t, we use a right hand rule with fingers curling in the orientation of t, we label Δ_σ with a '+' if the thumb points into Δ and a '-' if the thumb points out of Δ. With the additional orientation of Δ, we may read off the labels from nothing more than the gluing maps in $\mathbf{\Phi}$.

A disc d in a 3-simplex Δ is a *normal disc* if $d \cap \Delta_\sigma$ is either empty or a single normal arc for each face σ of Δ (the usual normal triangles and normal quadrilaterals). A surface S properly embedded in M is a *normal surface* if the lift of $S \cap \Delta$ is a collection of disjoint normal discs for all $\Delta \in \mathbf{\Delta}$. Let α be an edge (normal arc) in the interior of the induced cell-decomposition of S lying in a 2-simplex t. Being interior, α is the edge of exactly two normal discs d_1, d_2 in S lying in (possibly non-distinct) tetrahedra Δ_i, Δ_j common to t. If the faces of Δ_i and Δ_j are labeled one '+' and

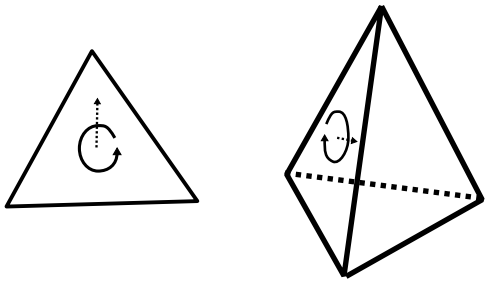


FIGURE 2. Orientations of a 2- and a 3-simplex

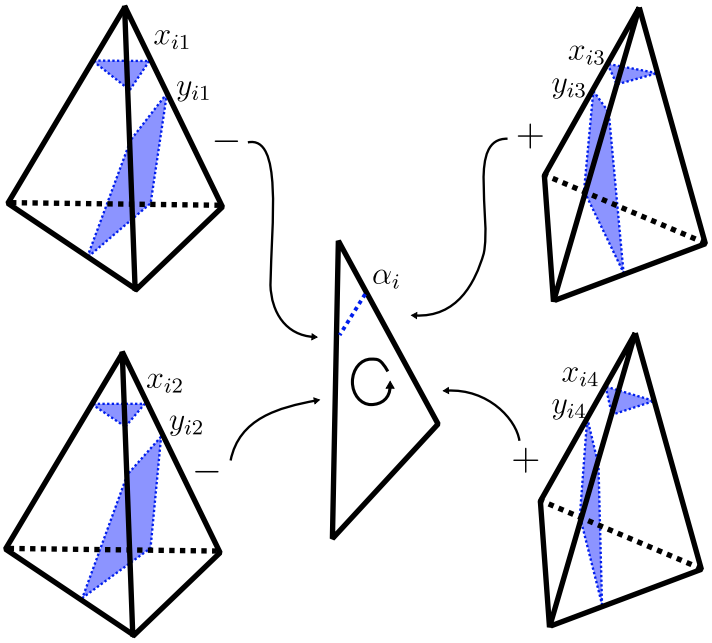


FIGURE 3. A collection of normal disc isotopy classes in 3-simplices labeled '+' and '-' all meeting at the central face.

the other '-' then we say S is *t(ransverse)-oriented* at α. If S is t-oriented for all choice of normal arc α then we say S is t-oriented. The induced cell-decomposition of normal surfaces has a 1-skeleton that is a normal graph in the 2-skeleton of the 3-complex. *Any knowledge gained from that area could help here.*

Let S be a t-oriented normal surface in M and t a face in M with associated +,-labeled 3-simplices. Let α_i be a normal arc class in t. We associate a homogeneous equation to α_i

$$\sum x_{ij} + \sum y_{ij} = \sum x_{ik} + \sum y_{ik}$$

where the x_{ij} are the number of normal triangles of S in a 3-simplex Δ_j meeting α_i in a face labeled +, y_{ij} are the number of normal quadrilaterals of S in Δ_j meeting α_i in a face labeled +, x_{ik}, y_{ik} are the number of normal triangles and quadrilaterals in a 3-simplex Δ_k meeting α_i in a face labeled '-'. See Figure 3. We associate to each normal arc class one such equation, the full system of homogeneous equations being called the *oriented matching equations*.

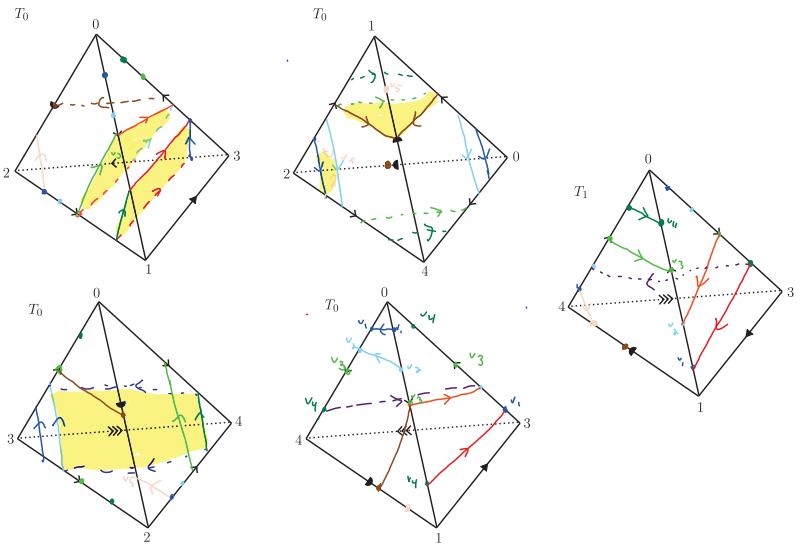


FIGURE 4. A t-oriented normal 2-sphere S_1. The edge labeling via arrows completely determine the face gluings.

After noting that in the case M is a triangulation or an ideal triangulation of a 3-manifold the oriented matching equations agree with the usual matching equations, the question then arises about general integral solutions to the system and their relation to normal surfaces. As in the setting of 3-manifold triangulations, we impose a quadrilateral constraint: for the three normal quadrilateral discs in a given 3-simplex, we require that at most one of the corresponding coordinates y_{ij} be non-zero; thus ensuring embedding. Now, given a solution to the oriented matching equations satisfying the quadrilateral constraint, does the solution correspond to a unique t-oriented normal surface? The answer is unfortunately, no. Compare Figures 4 and 5.

As the t-oriented normal surfaces in Figure 4 and 5 are distinct, there must be some way of distinguishing them, preferably taking the form of additional algebraic information. This can be achieved by extending Q-theory to our setting of 3-complexes. But first we point out the glaring difference: around the vertices of S_1 and S_2 we find neighborhoods intersecting distinct sequences of 3-simplices.

As in [2] we assign to the vertices of each normal quadrilateral a sign of ± 1 determined by the edges of the quadrilateral as it sweeps between the faces of Δ incident to the vertex. For a t-oriented normal surface we choose to always sweep from a face labeled '-' to a face labeled '+'. If both faces adjacent to the vertex have the same label, then the given quadrilateral cannot be included in any t-oriented normal surface. We call such a quadrilateral *non-admissible*; a quadrilateral is otherwise *admissible*

Given an edge ε in M, we associate with ε a directed graph with both vertices and edges labeled by the following procedure: for each edge in the lift of ε back to $\mathbf{F}$, we add an edge with label the name of the edge found in $\mathbf{F}$. If there exists a 3-simplex Δ containing ε in M such that two faces, one labeled '-' and the other '+', meet ε

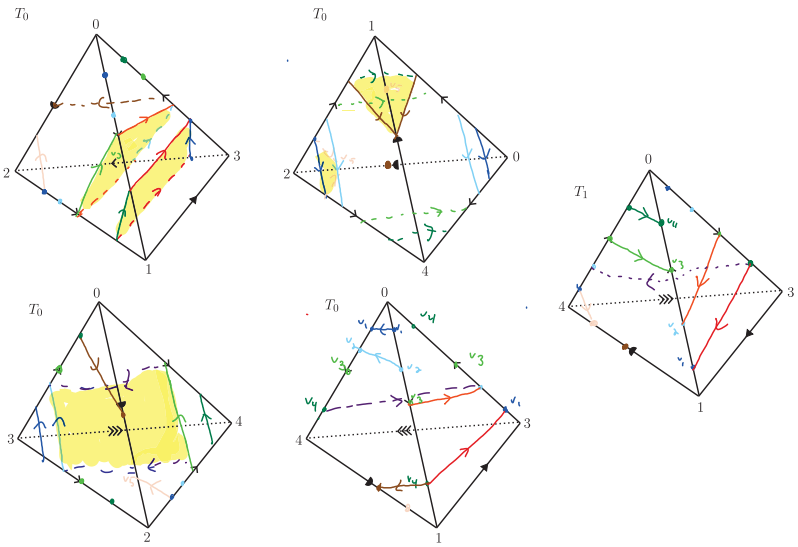


FIGURE 5. S_2, the disjoint union of a t-oriented normal 2-sphere and a t-oriented normal torus together consisting of the same normal discs as in Figure 4.

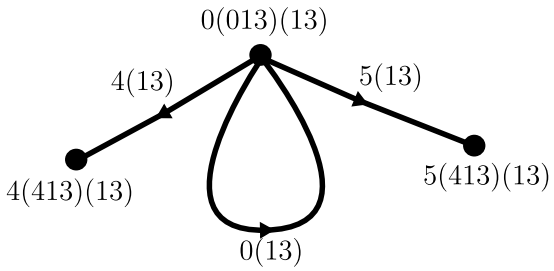


FIGURE 6. The oriented graph from Example 2.1.

then we add a directed edge oriented from the vertex associated to the edge in the '-' face to the vertex associated to the edge in the '+' face. We then label this directed edge with the name of the edge found in the lift of Δ.

Example 2.1. In Figures 4 and 5, faces $0(013), 0(213), 4(013), 5(013)$ are all identified with their respective edge (13) labeled with a solid arrow. We build a directed graph first with vertices $0(013)(13), 4(413)(13), 5(413)(13)$. Directed edges are added from 3-simplices $0, 4$, and 5. The graph is given in Figure 6. We can see immediately that, in this 3-complex, no t-oriented normal surface can meet faces $4(413)$ or $5(413)$.

The t-oriented normal surfaces can be enumerated by picking for each edge in M a number of cycles in the associated directed graph, one cycle for each desired distinct quadrilateral vertex neighborhood. To each cycle is associated a homogeneous equation

$$\sum s_j q_{ij} = 0$$

where q_{ij} is the number of admissible normal quadrilaterals of a given class incident to ε_i and $s_j \in \{\pm 1\}$ is the corresponding sign. For each family of cycles, the associated system of homogeneous equations is the *oriented Q-equations*.

The *link of a vertex* in M is the 2-complex corresponding to all normal triangle classes separating a given vertex. Each solution to a system of oriented Q-equations having at most one normal quadrilateral class per 3-simplex is associated to a family of t-oriented normal surfaces. These surfaces are constructed by first identifying the edges of any quadrilaterals that share a vertex and meet in the same face of M. The surfaces are completed by then taking all embedded subcomplexes of the vertex links of M whose un-identified edges pair with the un-identified quadrilateral edges. If edges of the added normal triangles are left un-identified, this process is included (infinitely if needed).

For a familiar comparison, spun-normal surfaces in ideal triangulations are solutions to oriented Q-equations with a predetermined choice of cycle per triangulation edge. After identifying the appropriate quadrilateral edges, an annulus (and possibly some discs) consisting of normal triangles is attached and this process is repeated infinitely as the surface spins around a neighborhood of the cone vertex. This introduces the possibility of analogous "spun-normal" t-oriented surfaces, even in the 3-skeleton of a closed compact triangulated 4-manifold. This is not surprising, as the boundary of a neighborhood of a vertex in a 4-manifold triangulation is a 3-sphere; said 3-sphere may contain triangulated "vertex-linking" surfaces around which to spin.

3. Generalizing and To-Do

Having moved the workload of constructing t-oriented normal surfaces to the oriented Q-equations, one finds that the orientation is not necessary (we don't have a natural way to write down general matching equations without the orientation) for working with Q-equations. We will allow two on the same +,- side of a face to be glued along an edge. Working in the 3-skeleton of a triangulated 4-manifold, though these two discs both meet the same face transversely we can still find a non-zero angle between them by taking a small 4-ball neighborhood of any intersection point and mapping back to $\mathbb{R}^4$.

We then build an undirected graph for each edge with labeled edges for any 3-simplex with adjacent faces meeting at the edge. For the Q-equations we take choices of cycles in the aforementioned graphs and for each cycle a choice of either orientation. The orientation is used to assign both the sign of the affected quadrilaterals as well as to build the Q-equation as in Section 3. Finally, for the solutions to these systems of Q-equations all possible vertex linking submanifolds are attached. This process returns the t-oriented normal surfaces as well as possible new normal surfaces.

The embarkation of this study was with the spirit of identifying knotted normal surfaces. As of this writing, the many normal surfaces identified in the 3-skeletons of triangulated 4-manifolds have all been unknotted (confirmed by construction of embedded compression bodies bounded by the normal surfaces). As examples grow with triangulation complexity, care must be taken to keep the search for knottedness feasible. There is an obvious advantage of working with normal surfaces: the complement

of a normal surface has an induced cell decomposition whose 2-cells are all contained in the 3-skeleton. Thus the normal surface and 2-skeleton of the triangulation can be used to directly construct the 2-skeleton of the surfaces complement. The 2-skeleton can then be used to compute $\pi_1(M^4 \backslash S)$, granting access to some useful 4-manifold invariants. For example, GAP should be able to determine relatively quickly if there is a surjective homomorphism $\pi_1(M^4 \backslash S) \to G$ where G is non-cyclic.

References

[1] W. Jaco and H. Rubenstein, "Layered-triangulations of 3-manifolds," *arXiv:math/0603601*, 2006.

[2] E. Kang, "Normal surfaces in non-compact 3-manifolds," *Journal of the Australian Mathematical Society*, vol. 78, pp. 305 – 321, 2005.