NOTES ON INCOMPRESSIBLE SURFACES WITH BOUNDARY PRESERVED UNDER CRUSHING

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1. Notes

Due to Walsh [4] if M is an atoroidal, acylindrical, irreducible, compact 3-manifold with torus boundary components, $\mathcal{T}$ an ideal triangulation of M with essential edges[1], then every properly embedded, two-sided, incompressible surface in M that is not a fiber of a surface bundle or a virtual fiber is isotopic to a spun-normal surface in $\mathcal{T}$. From this, one immediately thinks of two things. The first being can you build an ideal triangulation in which a fiber of a surface bundle IS isotopic to a spun-normal surface. This is covered in the drafted paper. The second is the implication that, because in any inflation triangulation a fiber will shrink to an isotopic normal surface, we have bounded surfaces in inflations that do not correspond to spun-normal surfaces in the ideal triangulation obtained by crushing along the boundary of the associated 3-manifold.

While working on my dissertation under Neil, I was trying to figure out exactly what crushing was doing to these surfaces I was finding in the inflation triangulations (particularly fibers). Let $\mathcal{T}^*$ be an ideal triangulation of M as above, ξ some frame of $\mathcal{T}^*$. For a spun-normal surface, S in $\mathcal{T}^*$ it is straightforward to construct a normal surface, S' in the inflation $\mathcal{T}_\xi$ so that, for those tetrahedra of $\mathcal{T}_\xi$ in one-to-one correspondence with those in $\mathcal{T}^*$, the quadrilaterals of S are in one-to-one correspondence with the quadrilaterals of S'.

First take the quadrilaterals of S in $\mathcal{T}^*$. Build the core by adding normal triangles for each abstract polyhedra around each edge of $\mathcal{T}^*$ as described by Kang [2]. We now have a cell-decomposition, $\mathcal{C}$, of a surface homeomorphic to S in $\mathring{M}$, and if we pick the right height to truncate the tetrahedra of $\mathcal{T}^*$ we will have a properly embedded (the quads are assumed to satisfy the tetrahedral constraint) surface in M given by the same cell-decomposition. The boundary $\mathcal{C}$ is given by edges in the triangulation of ∂M induced by truncation of the vertices. That is, edges in the vertex-linking surface of $\mathcal{T}^*$. Now we add normal triangles to $\mathcal{C}$ until the boundary abuts ξ. Call this extension $\mathcal{C}'$. While inflating $\mathcal{T}^*$ along ξ, extend $\mathcal{C}'$ exactly as for closed normal surfaces in the Boundary Slopes paper [1] except ignore those arcs in faces of $\mathcal{T}^*$ corresponding to $\partial\mathcal{C}'$. What we have is an extension of $\mathcal{C}'$ with unglued edges meeting the faces of the tetrahedra added during the inflation process. Faces which are associated to hexagonal faces of the induced cell decomposition of the complement of the normal boundary in $\mathcal{T}_\xi$. This cell-decomposition is homeomorphic to $\mathcal{C}'$ by the same cell-like agruments you made in [1]. We extend the unglued edges of $\partial\mathcal{C}'$ to the material boundary of $\mathcal{T}_\xi$ by adding normal triangles to the unglued edges.

[1] for each edge, e in $\mathcal{T}$, the arc $e \cap \mathring{M}$ is not homotopic relative its endpoints to $\partial\mathring{M}$

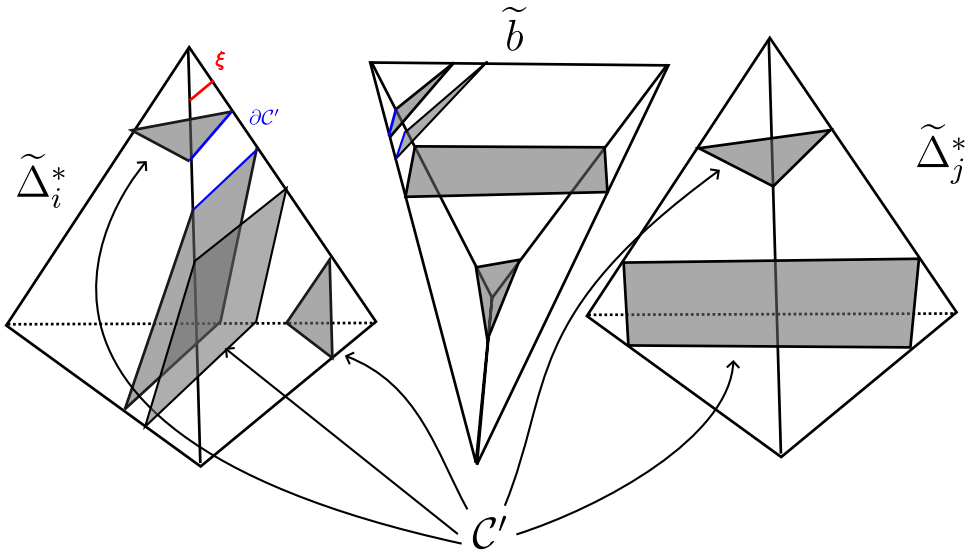


FIGURE 1. Extending $\mathcal{C}'$ in the inflation $\mathcal{T}_\xi$. Notation from Draft.

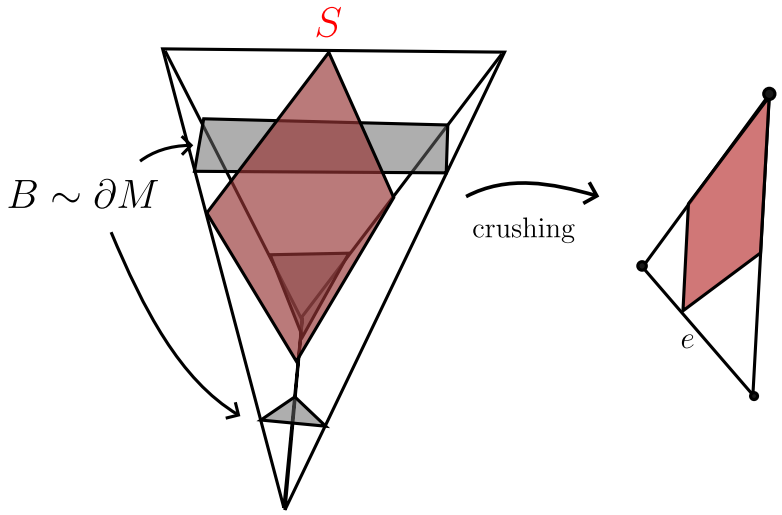


FIGURE 2.

See Figure 1. Then attach appropriate normal discs in the crossing and branch point tetrahedra to finish extending the surface to the boundary via an annulus.

If every spun-normal surface is extended this way, there must be something special about the normal triangles being the only disc types in the "band tetrahedra" that meet the material boundary of $\mathcal{T}_\xi$. So the new goal was to try and find some combinatorial structure after crushing for the surface if it instead had normal quadrilaterals meeting that boundary edge. The result I gave a talk at Fall '22 Redbud at UofArk, what I then called PCN-surface and now less affectionately call AC-surfaces (for *Artificium Confractiōnis*). One can track the surface through the crushing process and push it into the hexagonal face as shown in Figure 2.

AC-surfaces can be found in a similar manner to spun normal surfaces. Recall the notation for normal Q-theory [3] discussed briefly in the draft. We will modify the

Q-equations by adding additional senses. A given edge of ξ lying in a face σ of $\mathcal{T}^*$ separates and edge e of $\mathcal{T}^*$ from the linked vertex. Assign to the red quadrilateral in Figure 2 a sense $\epsilon_{i,\sigma}$ of $-1, +1$ for edge e and senses having opposing signs for the other two edges. We can add a variable to Tollefson's Q-matching equations for each such face σ:

$$\sum_{i=1}^{3n} \epsilon_{k,i} q_i + \sum_{\sigma} n_\sigma \epsilon_{i,\sigma} = 0 \tag{1.1}$$

This approach has two issues: 1) I have completely ignored the complications that arise when two or three edges of ξ lie in the same face of $\mathcal{T}^*$ (and this gets ugly). 2) we have just recreated (in part) the Q-matching equations for $\mathcal{T}_\xi$. Issue 2 suggests to give up on trying to resolve issue 1 and is the reason I have mainly dropped AC-surfaces and have begun looking exclusively at the Q-matching equations of a given inflation triangulation.

So, given a normal surface S in $\mathcal{T}_\xi$ with nonempty boundary (assume incompressible, ∂-incompressible or at the very least no component of ∂S is null homotopic in ∂M) assume S does *survive the crushing procedure*. That is, there is a spun-normal surface S' representing a properly embedded surface in M that is isotopic to S completely determined by the normal quads of S. Then S' satisfies the Q-matching equations of $\mathcal{T}^*$ (reminder that Q-theory is only known to work if each component of M is a torus). Take an edge, e in $\mathcal{T}^*$ and consider the edges in the lift of e through the crushing map. The Q-matching equations of $\mathcal{T}_\xi$ are satisfied by the normal quads in S. If we examine the senses of each quad in S to the respective lifts of e, it must be that they add collectively to zero. This is illustrated in Section 4.2 of my dissertation. This gives a necessary and sufficient condition for a normal surface in $\mathcal{T}_\xi$ to survive crushing (of course it agrees with your work in [1]). It is left to make appropriate arguments that the corresponding surfaces are homeomorphic. I went around this in the Draft by only looking at the fiber surviving crushing and the fact if you have a two-sided incompressible surface with nullhomologous boundary in a surface bundle over S^1, then the surface is isotopic to a fiber. I did this to save the full proof for this work.

2. Brief Applications

One application of this is to extend the work in the Draft. Being able to fully determine when a surface survives crushing avoids the headaches that get introduced when considering multi-punctured surface bundles over S^1 (the reason the Draft only treats knots). The ideas herein should also be readily applicable to immersed normal surfaces. This gets the rest of the missing surfaces from Walsh's result, namely virtual fibers which appear as immersed surfaces in virtually fibered 3-manifolds. So we should be able to get something along the lines of

Theorem 2.1. *Let M be an irreducible, ∂-irreducible, atoroidal, orientable, compact 3-manifold with boundary each component of which is homeomorphic to a torus.*

- *If M fibers over the circle with fiber F, then there is an ideal triangulation $\mathcal{T}^*$ of $\mathring{M}$ with a spun-normal surface S representing F.*

- *If there is a finite sheeted cover $p : \widetilde{M} \to M$ that fibers over the circle with fiber F, then there is an ideal triangulation $\mathcal{T}^*$ of $\mathring{M}$ with an immersed spun-normal surface S representing $p(F)$.*

There may need be some care with the quantifier on the fibers. There are link exteriors, L with an infinite family of properly embedded surfaces that are fibers of a bundle structure on L where the slopes of the surfaces are distinct. So, the above may need to be rephrased to "a spun-normal surface isotopic to a fiber of a bundle structure on M" or there abouts.

References

[1] B. Bryant, W. Jaco, and J. H. Rubinstein, "Efficient triangulations and boundary slopes," *Topology and its Applications*, vol. 297, p. 107689, 2021.

[2] E. Kang, "Normal surfaces in non-compact 3-manifolds," *Journal of the Australian Mathematical Society*, vol. 78, pp. 305 – 321, 2005.

[3] J. L. Tollefson, "Normal surface q-theory," *Pacific Journal of Mathematics*, vol. 183, pp. 359–374, 1998.

[4] G. Walsh, "Incompressible surfaces and spunnormal form," *Geometriae Dedicata*, vol. 151, pp. 221–231, 2005.